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PROBABILITY OF DEFECTIVE ASSEMBLIES WHEN COMPONENT TOLERANCES ARE INCORRECT

Lance W. Jayne, et al

Army Natick Laboratories
Natick, Massachusetts

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TECHNICAL REPORT
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FOREWORD

A major problem in all engineering design is that of reliability and quality control. Standard procedures exist for assuring the desired reliability under ordinary circumstances. This report describes an unusual situation, in which the tolerances on the components of an assembly are not small enough to ensure that the assembly will work properly, and provides estimates of the probability of a defective assembly in this case.

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ABSTRACT

This report describes an investigation of how errors in components of an assembly can affect its performance. In particular the report deals with the situation, uncommon in engineering practice, where the output tolerance of the assembly may be violated even though the tolerances on the components are all met. This situation is analyzed to estimate the probability that the output tolerance will be satisfied given that the component tolerances are met. Three methods are described for estimating this probability, their results are compared in a number of cases, and a best method is chosen. Several simple rules, suitable for preliminary estimates, are also given. An example is worked out showing a simple application of the method.

1. INTRODUCTION

This report deals with certain aspects of the general problem of errors and tolerances in the design and testing of equipment. It is presumed that the piece of equipment is required to operate at a certain level of output. Ordinarily the designer assigns a certain error-tolerance to this output, chosen so that the equipment will function properly if the output error satisfies its tolerance. The output error usually arises from errors in the various components that have been assembled to make the piece of equipment. The designer will customarily know the relation between the output error and the component errors. Common practice (see Bowker and Lieberman[1]) is that the designer will combine this relation with the output tolerance to find tolerances on each component such that satisfaction of these component tolerances will ensure that the output tolerance is met.

We are concerned here with the uncommon situation where satisfaction of the component tolerances does not ensure satisfaction of the output tolerance. This state of affairs can arise when an error has been made in choosing the component tolerances, or when it is impractical (or too expensive) to make the component tolerances small enough. In either case we must face the possibility that all the components will meet their tolerances but some of the assembled pieces of equipment will not work properly. The practical information that we want is the probability that the output tolerance will

be satisfied. With this information we can estimate how many extra pieces of equipment must be manufactured on the average in order to obtain a given number of workable assemblies.

In the following section we shall describe the general procedure for estimating the probability that the output satisfies its tolerance, supposing that each component error is normally distributed with zero mean, known variance and known tolerance. Three mathematical methods are given for carrying out the calculations. One is of Monte-Carlo type and is described in Section 3. The other two methods use the Characteristic Function in different ways. Section 4 gives formulas for the Characteristic Functions of the various distributions, and Sections 5 and 6 use these formulas in estimating the desired probability. Various simple approximations and limiting cases are examined in Section 7. Section 8 describes the results, which are then discussed in Section 9. The report closes with a simple example of how these estimates might be used in practice, Section 10.

We let Y_j be the error in the j-th component, $j = 1, 2, \ldots N$, and X_o is the error in the output. The relation between the output error and the component errors is taken as linear,

$$X_o = \sum_{j=1}^{N} C_j Y_j \tag{1}$$

where the C_j are assumed to be known constants.

It is assumed initially that Y_j is normally distributed with zero mean and variance σ_j^2. Then we may define

$$\left.\begin{aligned} X_j &= C_j Y_j \\ S_j &= |C_j|\sigma_j \end{aligned}\right\} \quad j = 1, 2, \ldots N \tag{2}$$

and the relation (1) can be written

$$X_o = \sum_{j=1}^{N} X_j \tag{3}$$

where X_j is normally distributed with zero mean and variance S_j^2.

We let $D_j > 0$ be the tolerance on the error, Y_j, in the j-th component, and $B_o > 0$ be the tolerance on the output error, X_o. Thus, when Y_j satisfies its tolerance, we have

$$|Y_j| \leq D_j$$

and, if X_o satisfied its tolerance, then

$$|X_o| \leq B_o$$

We define also

$$B_j = |C_j| D_j \tag{4}$$

as the tolerance on X_j, so that, if X_j satisfied its tolerance, then

$$|X_j| \leq B_j$$

We notice also that

$$B_j / S_j = D_j / \sigma_j \tag{5}$$

We now define certain probabilities. P_j is the probability that the j-th error satisfies its tolerance, i.e.,

$$P_j = \text{prob}\left[|Y_j| \leq D_j\right] = \text{prob}\left[|X_j| \leq B_j\right] \tag{6}$$

P_c is the probability that all component errors satisfy their tolerances. We assume that the component errors are independent of each other, and therefore

$$\begin{aligned} P_c &= \text{prob}\left[|X_j| \leq B_j \quad \text{for all} \quad j\right] \\ &= \prod_{j=1}^{N} P_j = P_1 P_2 \ldots P_N \end{aligned} \tag{7}$$

Further, we define

P_0 = prob [the output error satisfies its tolerance and all component errors satisfy their tolerances]

$$P_0 = \text{prob} \left[|X_0| \leq B_0 \text{ and } |X_j| \leq B_j \text{ for all } j \right] \tag{8}$$

The theorem on compound probability asserts that

$$P_0 = \text{prob} \left[|X_0| \leq B_0, \text{ given that } |X_j| \leq B_j \text{ for all } j \right] \times \text{prob} \left[|X_j| \leq B_j \text{ for all } j \right] \tag{9}$$

The probability that is of greatest practical interest is

$$P^* = \text{prob} \left[|X_0| \leq B_0, \text{ given that } |X_j| \leq B_j \text{ for all } j \right] \tag{10}$$

Then we can write (9) with the aid of (7) and (10) as

$$P^* = P_0 / P_c = P_0 / \prod_{j=1}^{N} P_j \tag{11}$$

Finally, it is useful to define

$$\Delta = \sum_{j=1}^{N} B_j = \sum_{j=1}^{N} |C_j| D_j \tag{12}$$

If all the components satisfy their tolerances, then, using a familiar property of inequalities, we find

$$|X_0| = \left| \sum_{j=1}^{N} X_j \right| \leq \sum_{j=1}^{N} |X_j| \leq \sum_{j=1}^{N} B_j$$

and so, because of (12), X_0 must satisfy the inequality

$$|X_0| \leq \Delta \tag{13}$$

If $\Delta \leq B_o$ then (13) implies

$$|X_o| \leq \Delta \leq B_o$$

In this case we see that, if each component satisfies its tolerance, the output error X_o must always satisfy its tolerance, and from (10) and (11) we conclude that

$$P^* = 1, \quad P_o = P_c$$

This case is the common one in design practice, i.e., the tolerances are set so that, if each component meets its tolerance, the output will necessarily satisfy its tolerance. However, in this paper we are interested in the opposite case, where

$$\Delta > B_o$$

and

$$0 \leq P^* \leq 1$$

Our main objective is to estimate P^*. We define $F^*(X_o)$ as the density function of the output when the separate component errors all satisfy their tolerances. Since the component errors are normally distributed, their density functions, when they satisfy their tolerances, are symmetrically-truncated normal distributions, and $F^*(X_o)$ is the density function of a finite sum of such distributions. P^* is the integral between $-B_o$ and B_o of $F^*(X_o)$. Unfortunately $F^*(X_o)$ is not easily expressible in terms of the parameters

of the component density functions. However, we can make a number of simple comments about the behavior of $F^* (X_o)$.

(i) If the component error tolerances are all very large, i.e., $B_j >> S_j$, each component error is approximately normally distributed, hence $F^* (X_o)$ is approximately a normal function.

(ii) If N, the number of error components, is large, the Central Limit Theorem leads us to expect that F^* will be approximately a normal function.

(iii) Contrariwise, F^* will depart furthest from normality when some B_j/S_j are small and when N is small. This will be particularly so when one component dominates all the rest, so that effectively N = 1.

3. MONTE-CARLO METHOD

This method is based on counting the numbers of successes in sampling from distributions that are random, independent and normally distributed with zero mean. A computer program was written to carry out this procedure.

The program has as input the quantities C_j, D_j ($j = 1, 2, \ldots N$) and B_o, together with the list of random numbers. L sets of N random numbers are read in successively. For each set, the N random numbers are taken as the values of Y_j ($j = 1, 2, \ldots N$), and the value of X_o is calculated from (1). Counts are made of the following quantities:

n_j ($j = 1, \ldots N$) is the number of cases for which $|Y_j| \leq D_j$

n_c is the number of cases for which $|Y_j| \leq D_j$ for all j.

n_o is the number of cases for which $|Y_j| \leq D_j$ for all j and $|X_o| \leq B_o$

Then we obtain the estimates

$P_j = n_j/L \qquad j = 1, 2, \ldots N$

$P_c = n_c/L$

$P_o = n_o/L$

$P^* = P_o/P_c = n_o/n_c.$

In using this method we had to choose L large enough so that reasonably stable estimates of P* were obtained. The choice L = 200 was used, and the entire procedure repeated four times with different sets of random numbers. The final estimates of the probabilities are given as the means of the results for the four repetitions.

4. CHARACTERISTIC FUNCTIONS

The remaining two methods of estimating P* employ the Characteristic Function (or Fourier Transform) as the main tool in the analysis. In this section we present the general formulas that form the basis of these methods.

If F (X) is a density function, then

$$\phi(t) = \int_{X=-\infty}^{X=+\infty} e^{-itX} F(X)\, dX \tag{14}$$

is its Characteristic Function or Fourier Transform. It is unnecessary to dwell on the properties of the Characteristic Function which are well-known. We record only one formula, which is easily derived from the Complex Inversion Relation,

$$\int_{\alpha=-X}^{X} F(\alpha)\, d\alpha = \pi^{-1} \int_{t=-\infty}^{\infty} \phi(t)\, t^{-1} \sin(tX)\, dX \tag{15}$$

This formula expresses the area under the density curve between -X and X (or the probability that the variable lies between -X and X) directly as an integral of $\phi(t)$.

We define F_j (X_j) as the density function for X_j and $\phi_j(t)$ as the corresponding Characteristic Function. Similarly $\phi_o(t)$ is the Characteristic Function of the output distribution, F* (X_o). Since the X_j are assumed to be independent, it is well-known that

$$\phi_o(t) = \prod_{j=1}^{N} \phi_j(t) \tag{16}$$

Both methods of estimating P* are based on finding $\phi_o(t)$ from the $\phi_j(t)$ by means of (16).

To find $\phi_j(t)$ we first write down the density function of X_j, which is that for a normal distribution with variance S_j^2, truncated at $\pm B_j$,

$$F_j(X_j) = \left\{ \int_{u=-B_j}^{B_j} e^{-u^2/(2S_j^2)}\, du \right\}^{-1} e^{-X_j^2/(2S_j^2)}, \qquad |X_j| \leq B_j$$

$$= 0 \qquad |X_j| > B_j$$

This is then substituted into (14), and we find, after some manipulation,*

$$\phi_j(t) = \frac{e^{-\rho_j^2}}{\operatorname{erf} \gamma_j} \operatorname{Re}\left\{ \operatorname{erf}(\gamma_j + i\rho_j) \right\} \tag{17}$$

where

$$\gamma_j = B_j/(S_j\sqrt{2}) > 0 \tag{18}$$

$$\rho_j = t S_j/\sqrt{2} \tag{19}$$

A series representation of $\phi_j(t)$ in real terms may be derived by expanding the Error Function in a Taylor Series about γ_j. By means of (A.5) we find

$$\phi_j(t) = e^{-\rho_j^2}\left\{ 1 - \frac{2\pi^{-1/2} e^{-\gamma_j^2}}{\operatorname{erf} \gamma_j} \sum_{n=1}^{\infty} (-\rho_j^2)^n \frac{H_{2n-1}(\gamma_j)}{(2n)!} \right\} \tag{20}$$

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*For completeness a list of the basic formulas relating to the Error Function is given in Appendix A.

where H_K is the Hermite Polynomial of degree k. This series converges absolutely for any finite values of ρ_j and γ_j. Further, we may expand $e^{-\rho_j^2}$ about $\rho_j = 0$ and obtain explicitly the leading terms (up to t^4) in the expansion of $\phi_j(t)$ about $t = 0$,

$$\phi_j(t) = 1 - S_j^2\left(1 - \frac{2\pi^{-1/2}\gamma_j e^{-\gamma_j^2}}{\operatorname{erf}\gamma_j}\right)(t^2/2!) + S_j^4\left\{3 - 2\pi^{-1/2}\frac{\gamma_j(3+2\gamma_j^2)e^{-\gamma_j^2}}{\operatorname{erf}\gamma_j}\right\}(t^4/4!) - \cdots \tag{21}$$

Although the series in (20) converges for any finite values of ρ_j and γ_j, the convergence is slow when ρ_j is large, and an alternative method of computation is needed. For this purpose it is convenient to use the real and imaginary parts, W_r and W_i, of the complex function, W, defined in (A.9). From (17), (A.10) and (A.11) we obtain the exact formula

$$\phi_j(t) = \frac{1}{\operatorname{erf}\gamma_j}\left\{e^{-\rho_j^2} - e^{-\gamma_j^2}\left[W_r(\rho_j + i\gamma_j)\cos(B_j t) - W_i(\rho_j + i\gamma_j)\sin(B_j t)\right]\right\} \tag{22}$$

A rational approximation for W is given in (A.12), and from it we may derive the following approximations for W_r and W_i:

$$W_r(\rho_j + i\gamma_j) = \sum_{K=1}^{3} r_K\{(-\gamma_j\alpha_{Kj} + \rho_j B_j t)/(\alpha_{Kj}^2 + B_j^2 t^2)\} \tag{23}$$

$$W_i(\rho_j + i\gamma_j) = \sum_{K=1}^{3} r_K\{(\rho_j\alpha_{Kj} + \gamma_j B_j t)/(\alpha_{Kj}^2 + B_j^2 t^2)\} \tag{24}$$

Here

$$\alpha_{Kj} = \rho_j^{\,2} - \gamma_j^{\,2} - \eta_K , \tag{25}$$

and r_K and η_K are constants of the approximation, listed in Appendix A.

5. APPROXIMATION USING MOMENTS

This method consists of assuming that the output density function is approximately of the form

$$g(X_o) = S^{-1}(2\pi)^{-1/2}\left[G_o + G_2(X_o/S)^2 + G_4(X_o/S)^4\right] e^{-X_o^2/(2S^2)} \tag{26}$$

where G_o, G_2 and G_4 are constants to be determined, and S^2 is the exact second moment of the output. The constants G_o, G_2 and G_4 are chosen by matching moments, i.e., by using

$$\int_{-\infty}^{\infty} X_o^{2K} g(X_o)\, dX_o = M_{2K}, \qquad K = 0, 1, 2 \tag{27}$$

where M_{2K} are the exact, even-ordered moments of the output distribution. M_o, M_2 and M_4 are determined by using the well-known relation

$$M_{2K} = (-i)^{2K} \frac{d^{2K}\Phi_o}{dt^{2K}}(0) \tag{28}$$

Because the $\Phi_j(t)$ are characteristic functions and are all even in t, we have

$$\Phi_j(0) = 1, \quad \Phi_j'(0) = \Phi_j'''(0) = 0 \tag{29}$$

Differentiating (16) and using (29) we find

$$M_0 = \phi_0(0) = 1 \tag{30}$$

$$M_1 = -i\,\phi_0'(0) = 0 \tag{31}$$

$$M_2 = -\phi_0''(0) = \sum_{j=1}^{N}\left[-\phi_j''(0)\right] \tag{32}$$

$$M_3 = i\,\phi_0'''(0) = 0 \tag{33}$$

$$M_4 = \phi_0^{IV}(0) = 3\left[\phi_0''(0)\right]^2 + \sum_{j=1}^{N}\left[\phi_j^{IV}(0) - 3\{\phi_j''(0)\}^2\right] \tag{34}$$

From (21) we obtain

$$-\phi_j''(0) = S_j^2\{1 - [Z_j A'(Z_j)/A(Z_j)]\} \tag{35}$$

$$\phi_j^{IV}(0) = S_j^4\{1 - [Z_j A'(Z_j)/A(Z_j)](3 + Z_j^2)\} \tag{36}$$

where

$$Z_j = B_j/S_j = 2^{1/2}\gamma_j \tag{37}$$

and the functions A and A' are defined in (A.2) and (A.6). We calculate M_2 and M_4 by inserting these values in (32) and (34).

If we combine (26) with (27) and use the general formula

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} v^{2n} e^{-v^2/(2S^2)}\,dv = S^{2n+1}\prod_{k=1}^{n}(2k-1), \quad n \geq 1 \tag{38}$$

we obtain the following linear, algebraic equations for

G_0, G_2 and G_4:

$$G_0 + G_2 + G_4 = 1$$

$$G_0 + 3G_2 + 15G_4 = M_2/S^2 = 1 \tag{39}$$

$$3G_0 + 15G_2 + 105G_4 = M_4/S^4 = M_4/M_2^2$$

This system is solved for G_0, G_2 and G_4, the results are inserted into (26), and we can then calculate the approximate P* by means of

$$P^* = \int_{-B_0}^{B_0} g(X_0)dX_0 = 2\int_0^{B_0} g(X_0)dX_0$$

Using the general relation (38) again we find the following formula for P*

$$P^*(Z) = A(Z) - HZ(3 - Z^2)A'(Z) \tag{40}$$

where

$$H = \{3 - (M_4/M_2^2)\}/24 \tag{41}$$

$$Z = B_0/S \tag{42}$$

We shall call the estimate of P* given by (40) the moment approximation.

6. NUMERICAL INTEGRATION OF THE CHARACTERISTIC FUNCTION

This procedure consists merely of carrying out the integration of (15), i.e., evaluating

$$P^* = \int_{-B_0}^{B_0} F^*(X_0)\,dX_0 = (2/\pi)\int_0^\infty \phi_0(t)\, t^{-1} \sin(B_0 t)\, dt \tag{43}$$

where $\phi_0(t)$ is calculated from $\phi_j(t)$ by (16). The $\phi_j(t)$ are evaluated by use of (20) when ρ_j is of moderate size and (22) - (25) when ρ_j is large.

In general it is necessary to evaluate the integral by numerical means. To carry this out with sufficient accuracy is sometimes difficult because $\phi_0(t)$ dies away in oscillatory fashion as $t \to \infty$. Usually, most of the contribution to the value of the integral comes from near t = 0, but significant contributions can also come from further out, where $\phi_0(t)$ may oscillate rapidly. In cases where a significant contribution comes from the region of rapid oscillation, the integration must be carried out with great care.

The following procedure was adopted for carrying out these integrals. A fundamental range of t say T, was chosen, roughly small enough so that

$$\frac{d}{dt}\left[t^{-1}\,\phi_0(t)\,\sin(B_0 t)\right]$$

has no more than 5 zeroes in the range $(n-1)T \le t \le nT$ for n = 1, 2, 3, .. M.

M is taken so large that the ranges beyond $n = M$ contribute negligibly to the integral. For each range of t the integral was evaluated by Gaussian Integration, and the total integral obtained by adding the results for all the ranges. Some experimentation was needed to find a suitable number of points to use in the Gaussian Integration and to determine how large M should be.

The computer program that carried out this evaluation of P^* was occasionally slow-running and was used primarily for spot-checking the results of the other methods.

7. SIMPLE APPROXIMATIONS AND LIMITING CASES

A number of obvious, simple approximations for P* can be derived, and we describe three of them briefly here.

(i) If all the component tolerances are very large, i.e., if

$$D_j/\sigma_j = B_j/S_j \gg 1 \qquad \text{for all } j$$

the distribution associated with each X_j is approximately normal with variance S_j^2. Then $F^*(X_o)$ is approximately a normal distribution with variance S_I^2, where

$$S_I^2 = \sum_{j=1}^{N} S_j^2 \tag{44}$$

Hence we may write this estimate of P* as a function of B_o in the form

$$P_I^*(B_o/S_I) = A(B_o/S_I) \tag{45}$$

(ii) A different approximation may be obtained if we assume that $F^*(X_o)$ is a normal distribution with variance S_{II}^2, truncated at points $|X_o| = \tau_o$ i.e.,

$$F^*(X_o) = 0 \qquad |X_o| \geq \tau_o$$

In this case we find the following estimate

$$P_{II}^*(B_o/S_{II}) = A(B_o/S_{II})/A(\tau_o/S_{II}), \quad B_o < \tau_o$$
$$= 1, \qquad B_o \geq \tau_o$$

The properties of the error function imply that

$$P_{II}^{*}(B_o/S_{II}) \geq P_{I}^{*}(B_o/S_I) \tag{46}$$

The accuracy of this approximation depends on how well F* (X_o) is approximated by a truncated normal distribution and how precisely we can estimate τ_o. There are two cases in which this approximation may be tolerably accurate. First, if one component, say the L-th, dominates the others, i.e., $S_L^2 >> S_j^2$, $j \neq L$ the output will be approximately that of the dominant component, X_L, which is a normal distribution truncated at $\pm B_L$. In this case we expect that $P_{II}^{*}(B_o/S_{II})$ with $\tau_o = B_L$, will be a fair approximation to P*. Second, if all the C_j are roughly equal, and all the Z_j are roughly equal to Z_A, say, then we expect that F* (X_o) will be approximately a normal distribution with variance S_I^2 truncated at

$$\tau_o = Z_A S_I$$

Hence in this case also $P_{II}^{*}(B_o/S_{II})$ should be a decent approximation to P*.

(iii) A third simple approximation is obtained by setting $G_4 = 0$ in (26) and choosing G_o and G_2 such that

$$\int_{-\infty}^{\infty} X_o^{2K} g(X_o)\, dX_o = M_{2K} \quad \text{for} \quad K = 0, 1$$

Then we get as an approximation for P* merely the first term of (42),

$$P_{III}^{*}(Z) = A(Z). \tag{47}$$

8. RESULTS

All the cases discussed here have four components, i.e., $N = 4$, and all have $\sigma_j = 1$, $j = 1, 2, 3, 4$. Eleven different cases were studied with various values for the D_j and C_j as shown in Table 1. For each case results were obtained in the form of graphs of P* as a function of B_0/S_I and are displayed in Figures 1-11. Each graph shows the mean and standard deviation of the four repetitions of the Monte-Carlo Method as well as the moment approximation for that case. The values of S_I, $M_2 = S^2$ and M_4 are listed in Table 2.

The method of integrating the Characteristic Function was used only at points where sizeable discrepancies were found between the Monte-Carlo results and the moment approximation. These points are shown on the appropriate graphs and compared with the other methods in Table 3.

Several additional graphs show comparisons among the moment method predictions for different cases. The comparison among Cases (I), (II) and (III) is shown in Figure 12, Cases (VI), (VII) and (VIII) in Figure 13 and Cases (IX), (X) and (XI) in Figure 14. Also, the simple approximation P_I* is shown in Figure 12, and P_{II}* in Figures 13 and 14 for relevant values of τ_0.

9. DISCUSSION

We shall first compare and comment on the results obtained by the various methods, then suggest procedures for estimating P* under various circumstances.

Figures 1 to 11 show that the agreement between the Monte-Carlo method and the moment-approximation is reasonably good in a general sense. We see from Table 3 that, when the results do not agree well, the integration of the characteristic function almost always agrees with the moment approximation.

Of the three methods one expects that integrating the characteristic function should be the most accurate. The Monte-Carlo method is usually thought to be somewhat inaccurate unless a very large number of samples is used, and the above results suggest that this is the case here. Procedures like the moment-approximation are fairly common in statistics and often give satisfactory accuracy. However, there are two theoretical defects of the moment-approximation here that are worth mentioning. First, the approximate density function g (X_0) is continuous (see Equation (26)) but the true density function, F* (X_0), is discontinuous. In fact

$$F^*(X_0) \equiv 0$$

when $X_0 > \Delta$. Second, g (X_0) is slightly negative for X_0 sufficiently large in many cases. Neither of these defects seems to cause serious errors in the estimate of P* for the cases studied here since the results agree well with the integration of the characteristic function. If serious errors are to arise, one would expect to find them when there is a single, dominant

component with a low tolerance on it, as in Case (VIII). Table 3 confirms this expectation, for we see that, when $B_O/S_I = .945$, the characteristic function and moment-approximation differ by .011, whereas the worst error observed in the other cases of Table 3 is only .001. However, even in this most unfavorable case the error in the moment-approximation is small enough to be unimportant.

Figure 12 shows how curves for P* change as the common tolerance value for the four components increases from $B_j/S_j = 1$ through 1.5 to 2. As we expect, the curves become lower and tend toward the normal curve, given by P_I*, with increasing component tolerances.

The effect on P* of an increasingly dominant component is displayed in Figures 13 and 14. Figure 13 shows the case where the increasingly dominant component has a smaller tolerance than the other components. As C_1 increases from 1 through 2 to 5, the curve of P* is raised toward the curve for P_{II}* truncated at $B_O/S_I = 1$, to which it must ultimately tend. In contrast Figure 14 shows what happens when the increasingly dominant component has a higher tolerance than the others. As C_1 increases from 1 through 2 to 5, the curve for P* is lowered toward the curve for P_{II}* truncated at $B_O/S_I = 2$, to which it ultimately tends.

When all the C_j are roughly the same, we may also inquire about the effect of changing the component tolerances but keeping the average component tolerance constant. Comparing Cases (II), IV) and (V) we see that this has

scarcely any effect on P*. In other words, when the C_j are roughly equal, the mean component tolerance has a considerable influence on P* (see Figure 12) but the variance in the component tolerances has negligible effect.

A reasonably extensive comparison of the simple approximations, (47), (48) and (49) with the more accurate calculations suggests the following as a rule:

(i) Use P_{II}* if one component dominates greatly.

(ii) Use P_I* if $B_j/S_j >$ 2 for all j.

(iii) Use P_{III}* if no one component dominates.

Use of this rule will give fair results, perhaps suitable for an initial estimate. The simplest accurate procedure is the moment-approximation, given by (42) - (44).

10. WORKED EXAMPLE

A certain piece of mechanical equipment is supposed to operate at a load of 900 lbs. Three components, standard items, are assembled to form this mechanism. Components 1 and 3 are springs and component 2 is an electrical switch. Their component errors, Y_1, Y_2 and Y_3, are related to the error in the output load by (1) where

C_1 = 240 lb/inch

C_2 = 15 lbs/volt

C_3 = 210 lb/inch

From information about the manufacture of the components we know that the distribution of their errors is roughly normal with zero mean and

#1: 80% are acceptable at a tolerance of .1-inch

#2: 70% are acceptable at a tolerance of 1 volt

#3: 92% are acceptable at a tolerance of .1-inch

From Tables of the normal distribution we find the standard deviation σ_j as follows:

#1: .8 acceptable corresponds to $.1 = 1.28\,\sigma_1$

$\sigma_1 = .0781$-inches

#2: .7 acceptable corresponds to $1 = 1.04\,\sigma_2$

$\sigma_2 = .962$ volts

#3: .92 acceptable corresponds to $.1 = 1.75\,\sigma_3$

$\sigma_3 = .0571$ inches

Then

$S_1 = 240 \times .0781 = 18.7$ lbs.

$S_2 = 15 \times .962 = 14.4$ lbs.

$S_3 = 210 \times .0571 = 12.0$ lbs.

The tolerances established on the components are

$D_1 = .15$ inches, $D_2 = 2$ volts, $D_3 = .07$ inches

and therefore from (4) and (12)

$B_1 = 36$ lbs, $B_2 = 30$ lbs, $B_3 = 14.7$ lbs.

$\Delta = 80.7$ lbs.

The tolerance on the load needed to activate the mechanism is

$B_0 = 30$ lbs.

Since $B_0 < \Delta$ we know that $0 < P^* < 1$, i.e., we know it is possible that each component will satisfy its tolerance but the tolerance on the load will be violated.

First we find a quick, rough estimate of P*. In order to determine which one of the simple estimates, (45) - (47), is best, we find from (37)

$$Z_1 = B_1/S_1 = 1.93\,, \quad Z_2 = 2.08\,, \quad Z_3 = 1.22$$

These are not all ≥ 2. Also none of the C_j dominates all the others. The rule stated at the end of Section 9 suggests, therefore, that we use P_{III}. To find P_{III} we need to find M_2 by means of (32) and (35). From the tables of the normal function we find

$$A(Z_1) = .946, \quad A(Z_2) = .962, \quad A(Z_3) = .778$$

$$A'(Z_1) = .120, \quad A'(Z_2) = .092, \quad A'(Z_3) = .379$$

Putting these values into (35), then combining the results with (32) we get

$$M_2 = 480, \quad S = 21.9$$

From (42) $Z = 30/21.9 = 1.37$. Using the estimate (47), the normal function table gives

$$P_{III}^* = A\,(1.37) = .829$$

The more accurate approximation (40) involves finding M_4 in addition to M_2. The calculation of M_4 by (34) and (36) leads to $M_4 = 625{,}000$. Putting this into (41) and combining with (40) we find

$$P^*\,(Z) = .829 - .006 = .823$$

Thus, if all components satisfy their tolerances, the probability that the mechanism will operate at a load between 870 and 930 lbs. is about .82. On the average, therefore, if we need 1,000 workable mechanisms, we should

expect to assemble about

$$1000/.823 = 1215$$

out of satisfactory components.

REFERENCES

1. Bowker, A. H., and Lieberman, G. J., "Engineering Statistics", Prentice Hall, Englewood Cliffs, N. J., 1959, Chapter 3.

2. Abramowitz, M., and Stegun, I. A., "Handbook of Mathematical Functions", National Bureau of Standards, Applied Math Series No. 55, Sixth Printing, November 1967.

APPENDIX A - FORMULAS RELATING TO THE ERROR FUNCTION

The following are fundamental formulas relating to the Error Function and are taken from Reference [2].

$$\operatorname{erf}(Z) = 2\pi^{-1/2}\int_0^Z e^{-U^2}\,dU \tag{A.1}$$

In (A.1) the integration may be carried out along any path in the complex U-plane connecting U=O and U=Z. An alternative definition is

$$A(Z) = \operatorname{erf}(2^{-1/2}Z) = (2\pi)^{-1/2}\int_{\lambda=-Z}^{Z} e^{-\lambda^2/2}\,d\lambda \tag{A.2}$$

We have also

$$\operatorname{erf}(-Z) = -\operatorname{erf}(Z) \tag{A.3}$$

$$\operatorname{erf}(\bar{Z}) = \overline{\operatorname{erf}(Z)} \tag{A.4}$$

where the bar denotes the complex conjugate.

Various derivatives of these quantities can be found from

$$\frac{d^{K+1}}{dZ^{K+1}}(\operatorname{erf} Z) = (-1)^K\, 2\pi^{-1/2} e^{-Z^2} H_K(Z), \quad K \geq 0 \tag{A.5}$$

$$A'(Z) \equiv dA/dZ = (2/\pi)^{1/2} e^{-Z^2/2} \tag{A.6}$$

where H_k (Z) is the Hermite Polynomial of k-th degree. These satisfy the relations

$$H_0(Z) = 1, \quad H_1(Z) = 2Z \tag{A.7}$$

$$H_{K+1}(Z) = 2Z\,H_K(Z) - 2K\,H_{K-1}(Z). \tag{A.8}$$

Also we list several formulas involving the complex function W.

$$W(Z) \equiv W_r(Z) + iW_i(Z) = e^{-Z^2}[1 - \mathrm{erf}(-iZ)] \tag{A.9}$$

or

$$\mathrm{erf}(Z) = 1 - W(iZ)e^{-Z^2} \tag{A.10}$$

and

$$W(\bar{Z}) = \overline{W(-Z)} \tag{A.11}$$

W (Z) may be found from the rational approximation

$$W(Z) = iZ\sum_{K=1}^{3} r_K/(Z^2 - \eta_K) \quad + \epsilon(Z) \tag{A.12}$$

provided $W_r > 3.9$ or $W_i > 3$. The error $\epsilon(Z)$ satisfied the inequality $|\epsilon(Z)| \leq 2\times10^{-6}$. The constants r_K and η_K have the following values

$$r_1 = .4613135, \quad r_2 = .09999216, \quad r_3 = .002883894$$

$$\eta_1 = .1901635, \quad \eta_2 = 1.7844927, \quad \eta_3 = 5.5253437.$$

TABLE 1

Case	C_1	C_2	C_3	C_4	D_1	D_2	D_3	D_4
I	1	1	1	1	1	1	1	1
II	1	1	1	1	1.5	1.5	1.5	1.5
III	1	1	1	1	2	2	2	2
IV	1	1	1	1	1	1.5	1.5	2
V	1	1	1	1	1	1	2	2
VI	1	1	1	1	1	1.5	1.5	1.5
VII	2	1	1	1	1	1.5	1.5	1.5
VIII	5	1	1	1	1	1.5	1.5	1.5
IX	1	1	1	1	2	1	1	1
X	2	1	1	1	2	1	1	1
XI	5	1	1	1	2	1	1	1

Component Error Coefficients (C_j) and Tolerances (D_j) in the Eleven Cases Studied

TABLE 2

Case	S_I	M_2	M_4
I	2	1.165	3.709
II	2	2.206	13.53
III	2	3.095	27.22
IV	2	2.168	13.10
V	2	2.130	12.67
VI	2	1.946	10.47
VII	2.646	2.819	21.60
VIII	5.292	8.933	182.5
IX	2	1.647	7.490
X	2.646	3.968	40.90
XI	5.292	20.22	988.5

The Standard Deviation, S_I, of the Output Distribution Calculated by the Normal Relation, and the Output Moments M_2, M_4, in the Eleven Cases Studied

TABLE 3

Case	B_o	B_o/S_I	P* Monte Carlo	P* Char. Function	P* Mom. App.
(I)	1.6	.8	.822	.858	.858
(III)	2.8	1.4	.860	.887	.887
(IV)	2.4	1.2	.870	.896	.896
(V)	1.2	.6	.603	.579	.580
(V)	2	1.0	.849	.825	.825
(VI)	1.2	.6	.612	.599	.600
(VIII)	5	.945	.928	.915	.904
(IX)	1.2	.6	.677	.639	.640
(IX)	2.4	1.2	.912	.940	.940
(X)	3.175	1.2	.907	.886	.886

Comparison among the Values of P* given by the Three Methods of Calculation for Various Cases and Tolerances

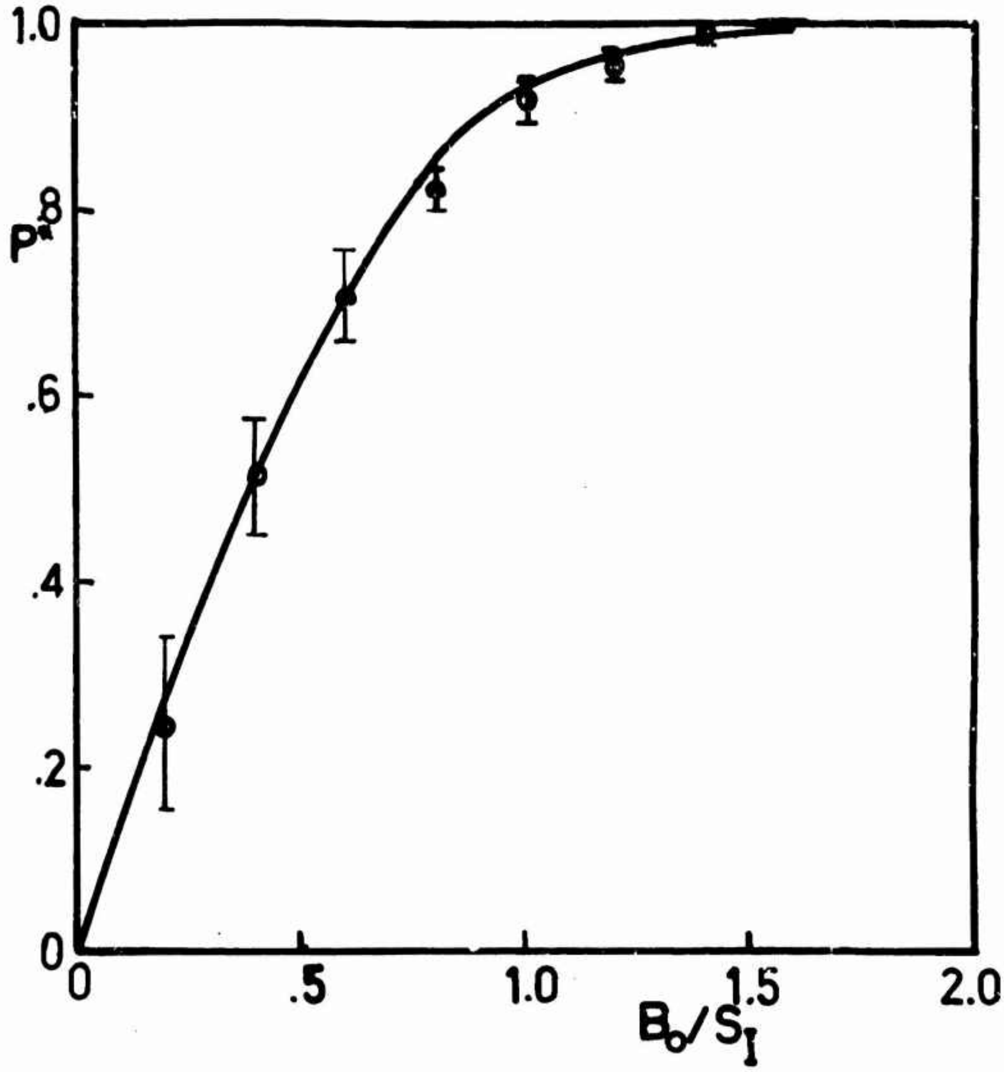

Figure 1: P* (B_o/S_I) in Case I. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

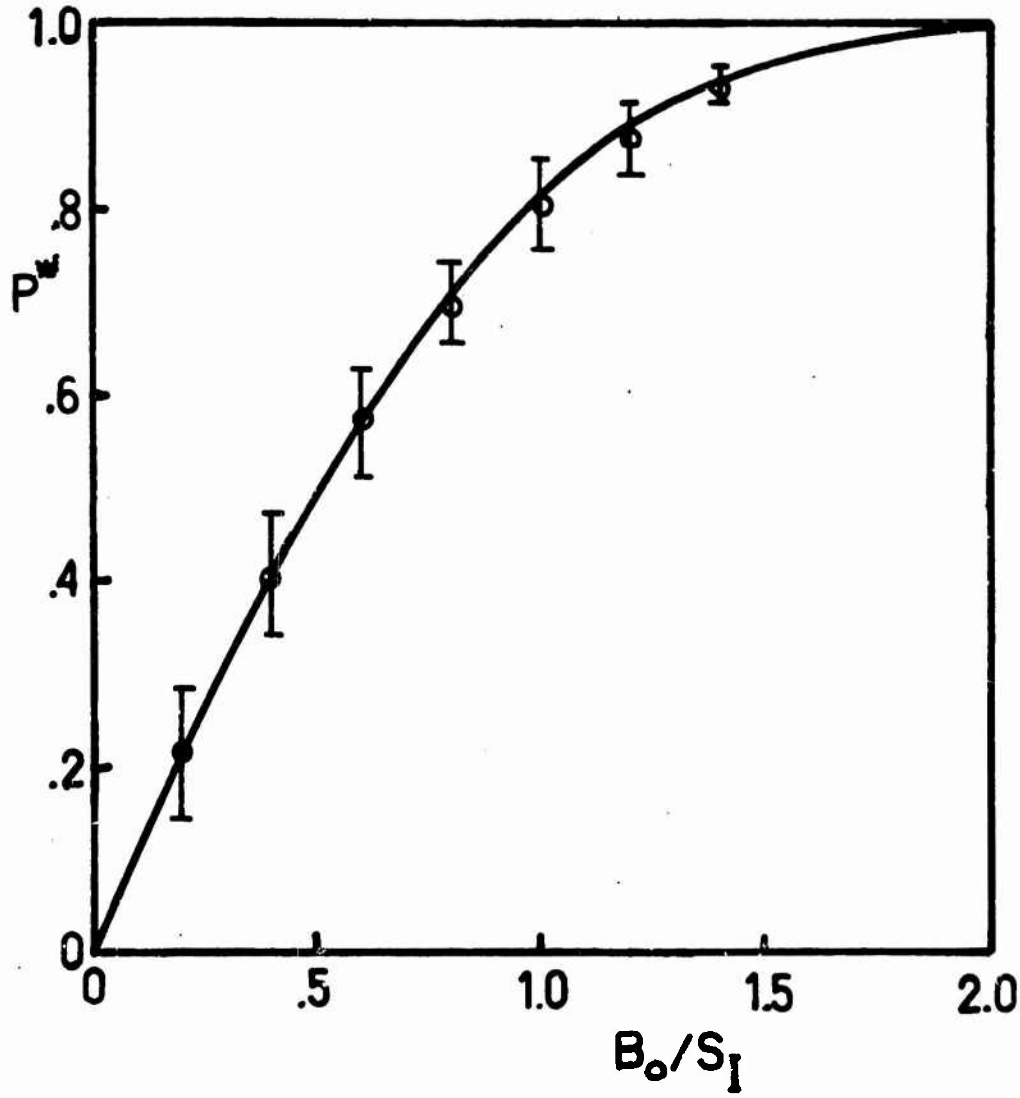

Figure 2: P* (B_o/S_I) in Case II. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

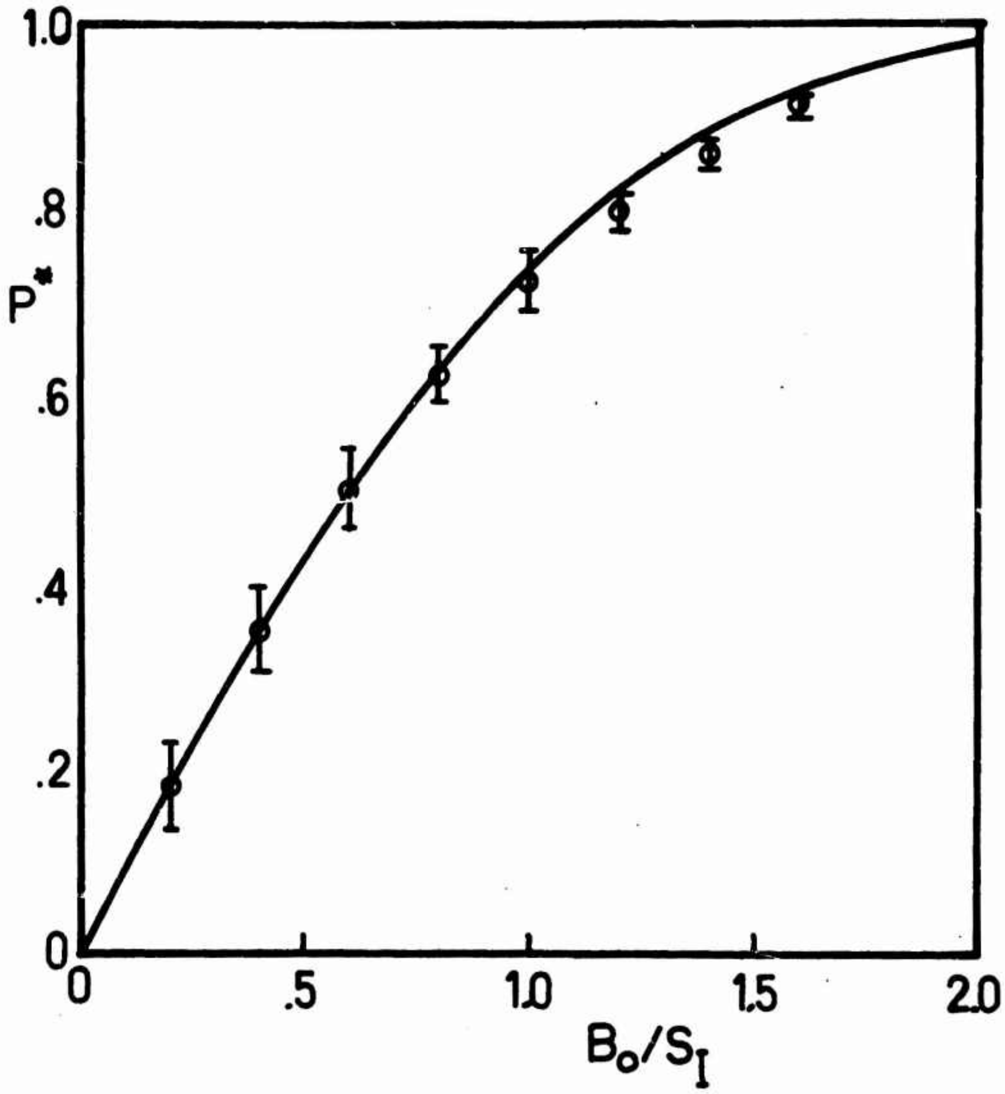

Figure 3: P^* (B_o/S_I) in Case III. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

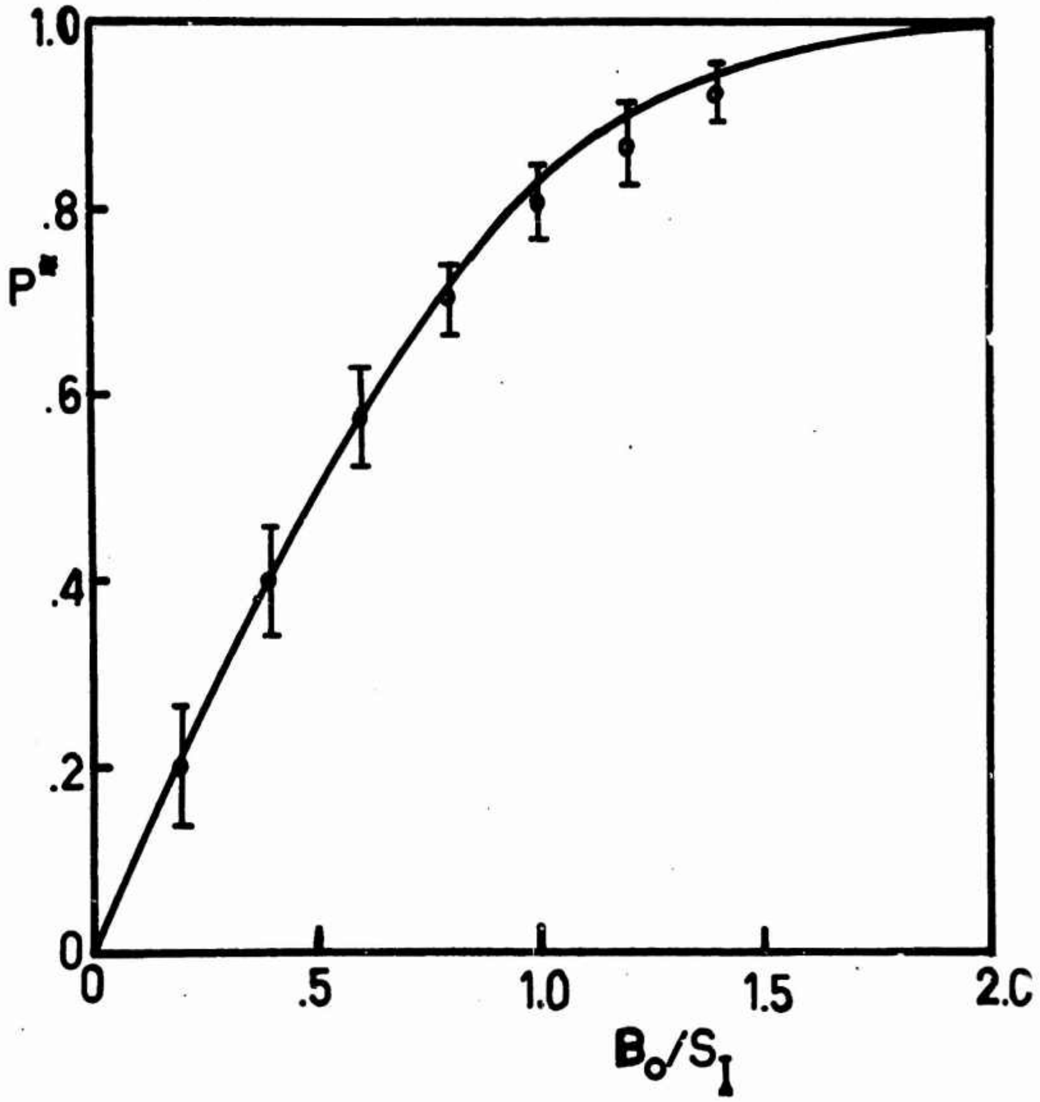

Figure 4: P* (B_o/S_I) in Case IV. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

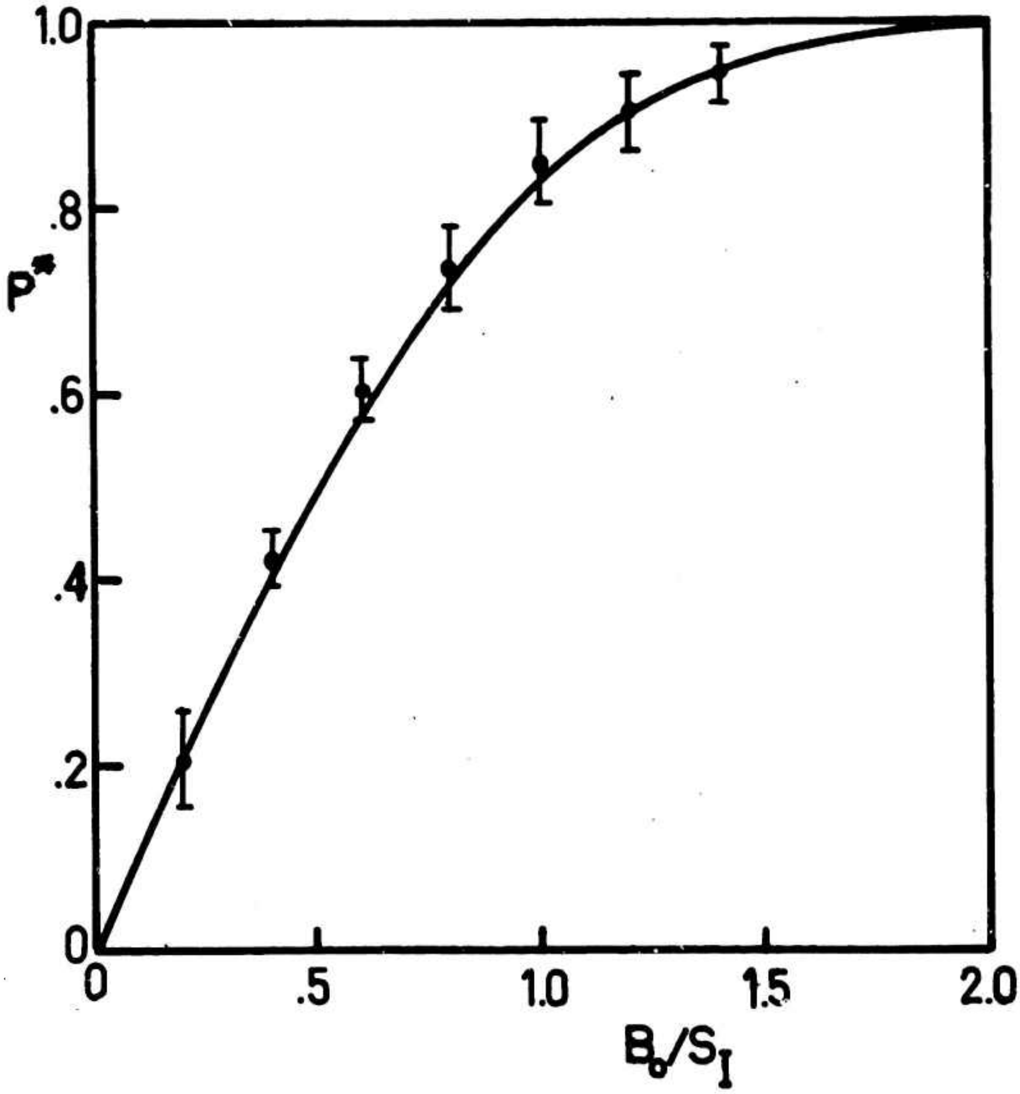

Figure 5: P* (B_0/S_I) in Case V. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

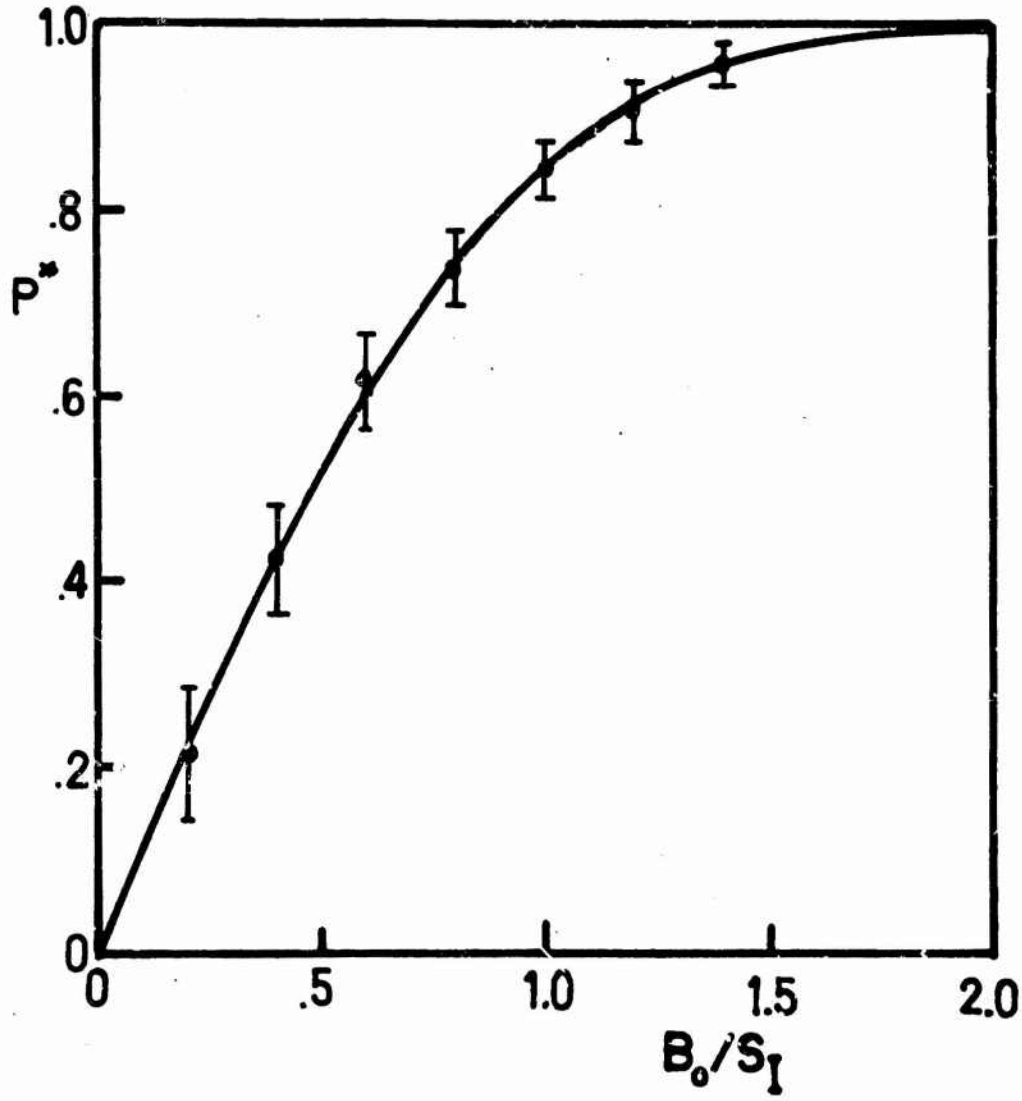

Figure 6: P* (B_o/S_I) in Case VI. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

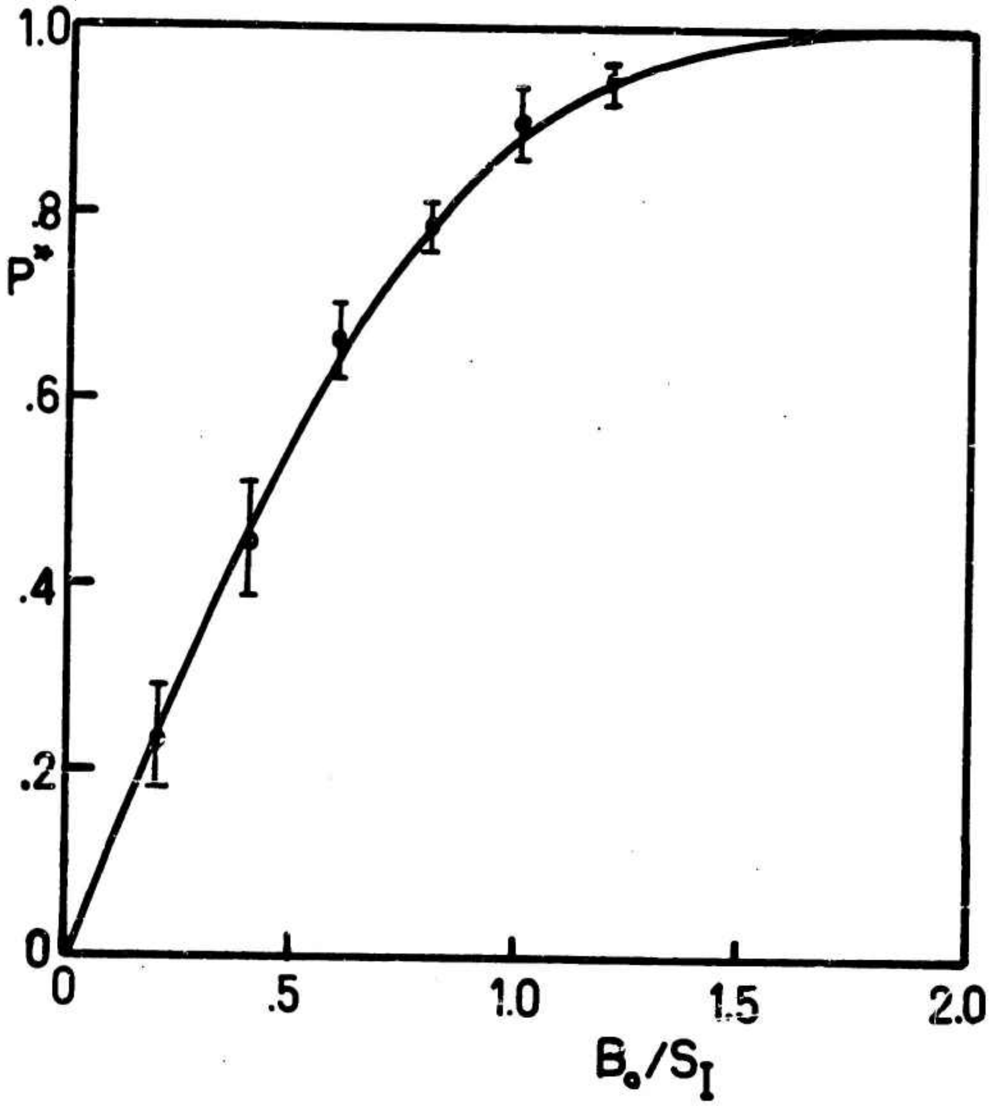

Figure 7: P* (B_O/S_I) in Case VII. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

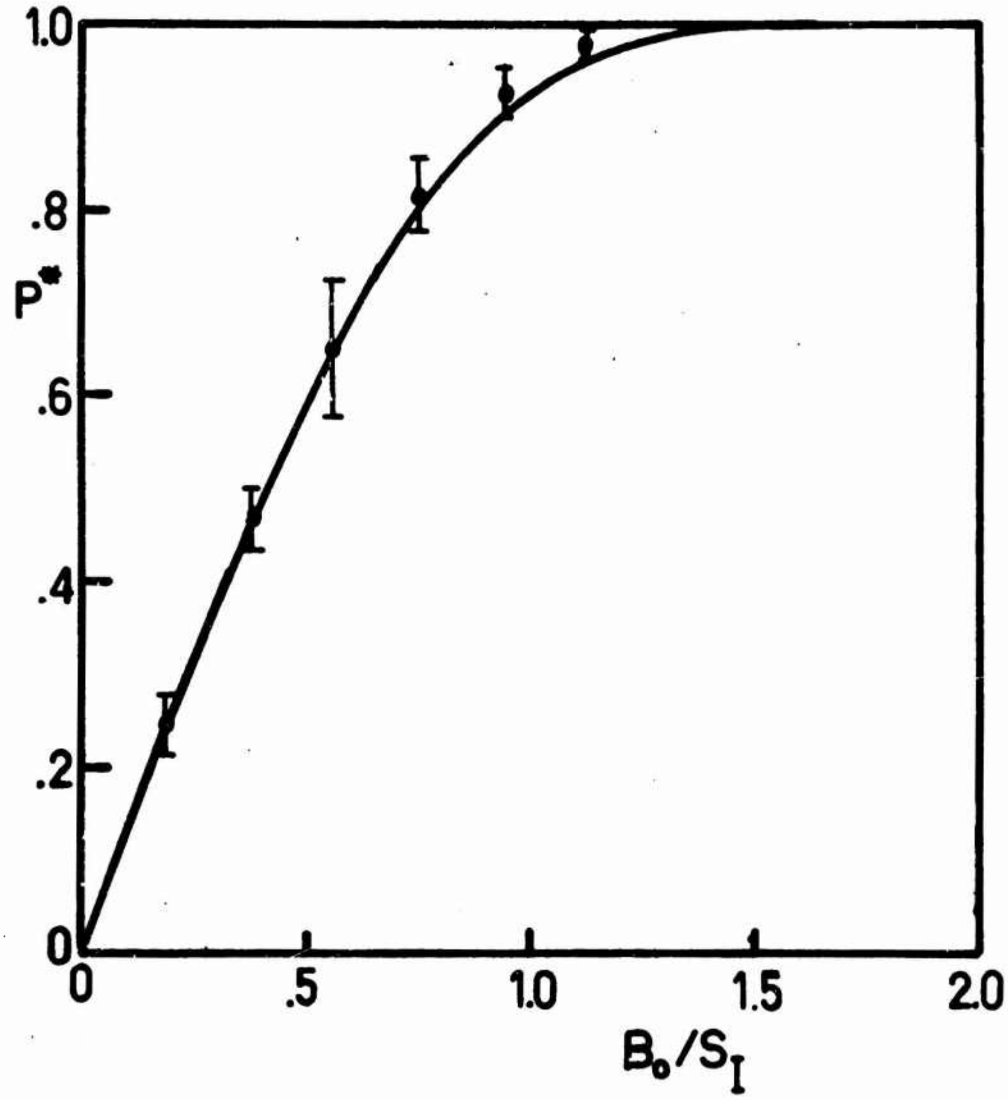

Figure 8: P* (B_o/S_I) in Case VIII. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

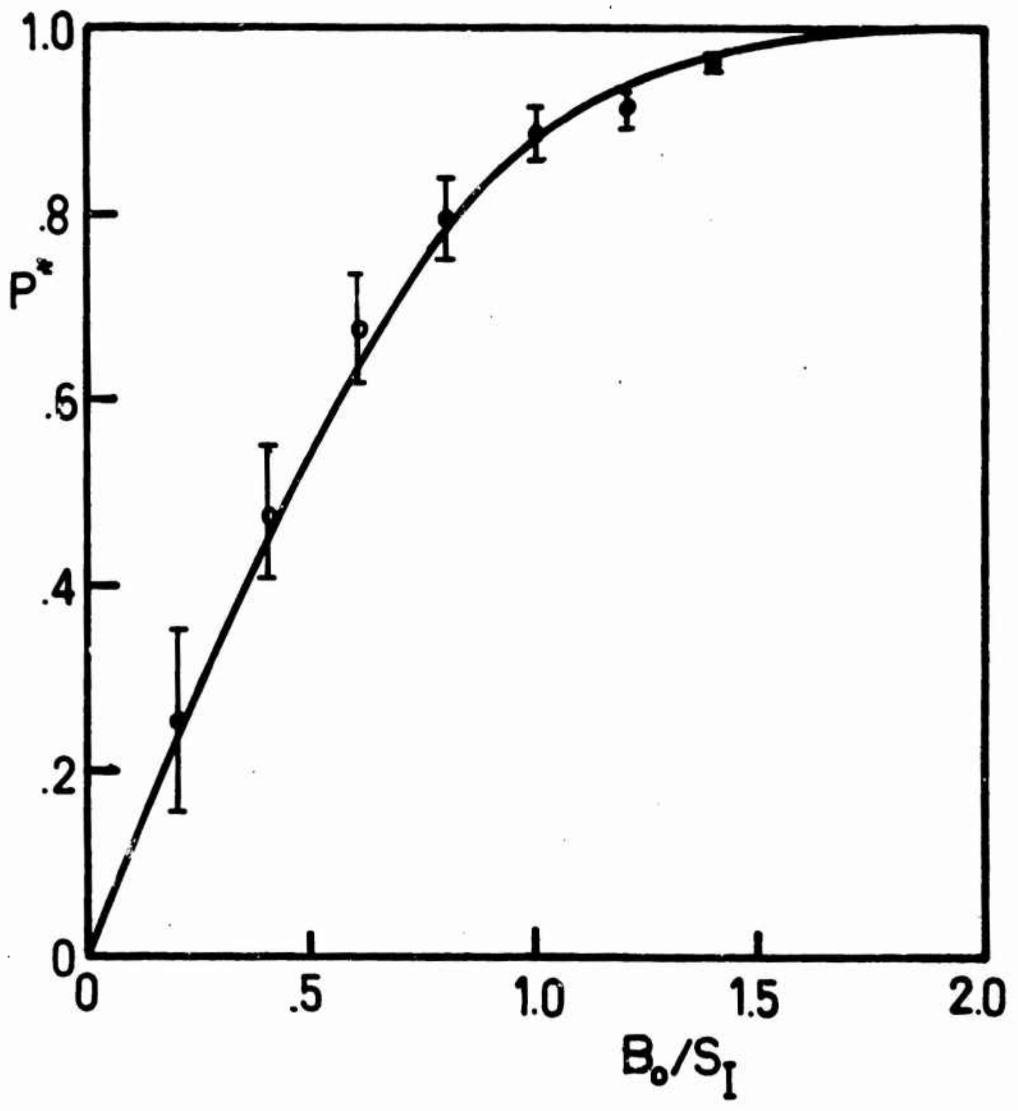

Figure 9: P* (B_O/S_I) in Case IX. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

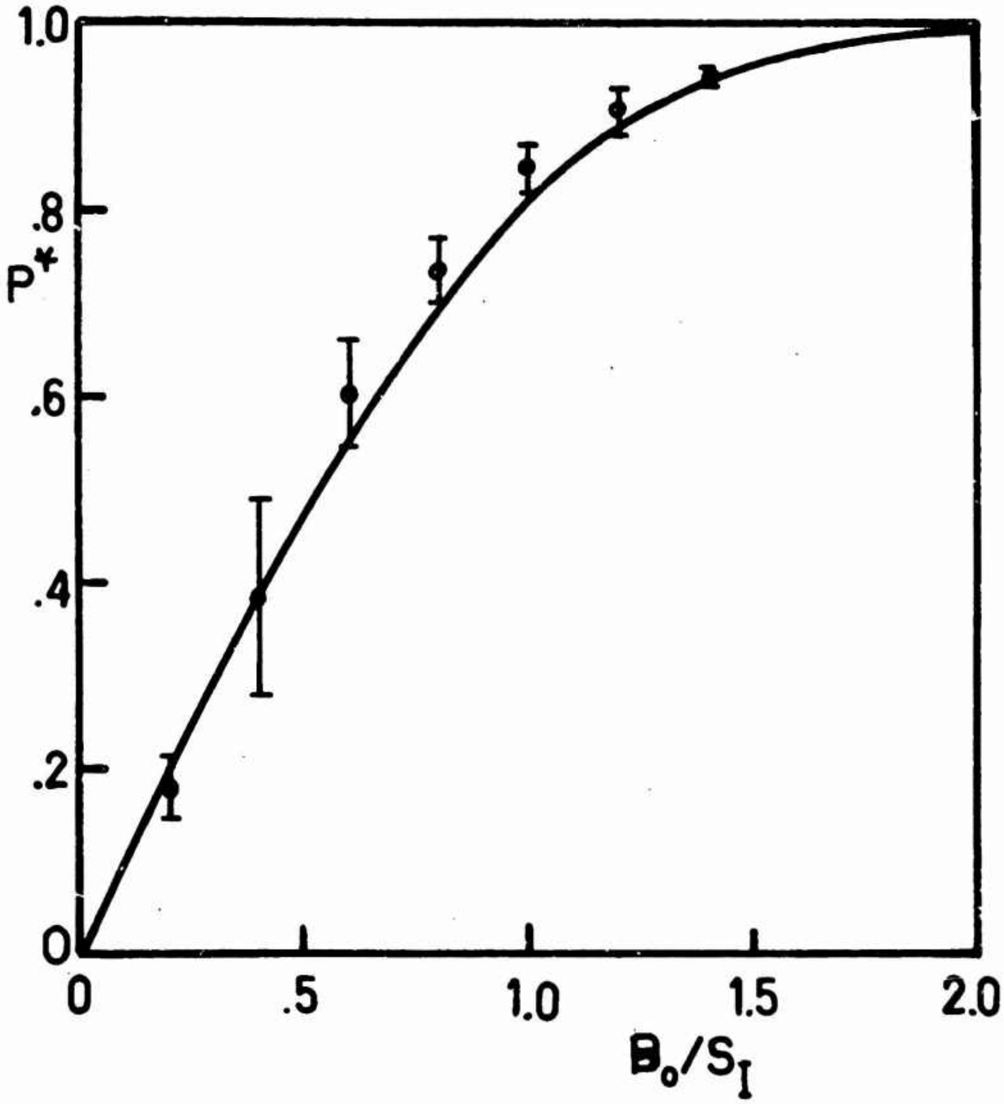

Figure 10: P* (B_o/S_I) in Case X. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

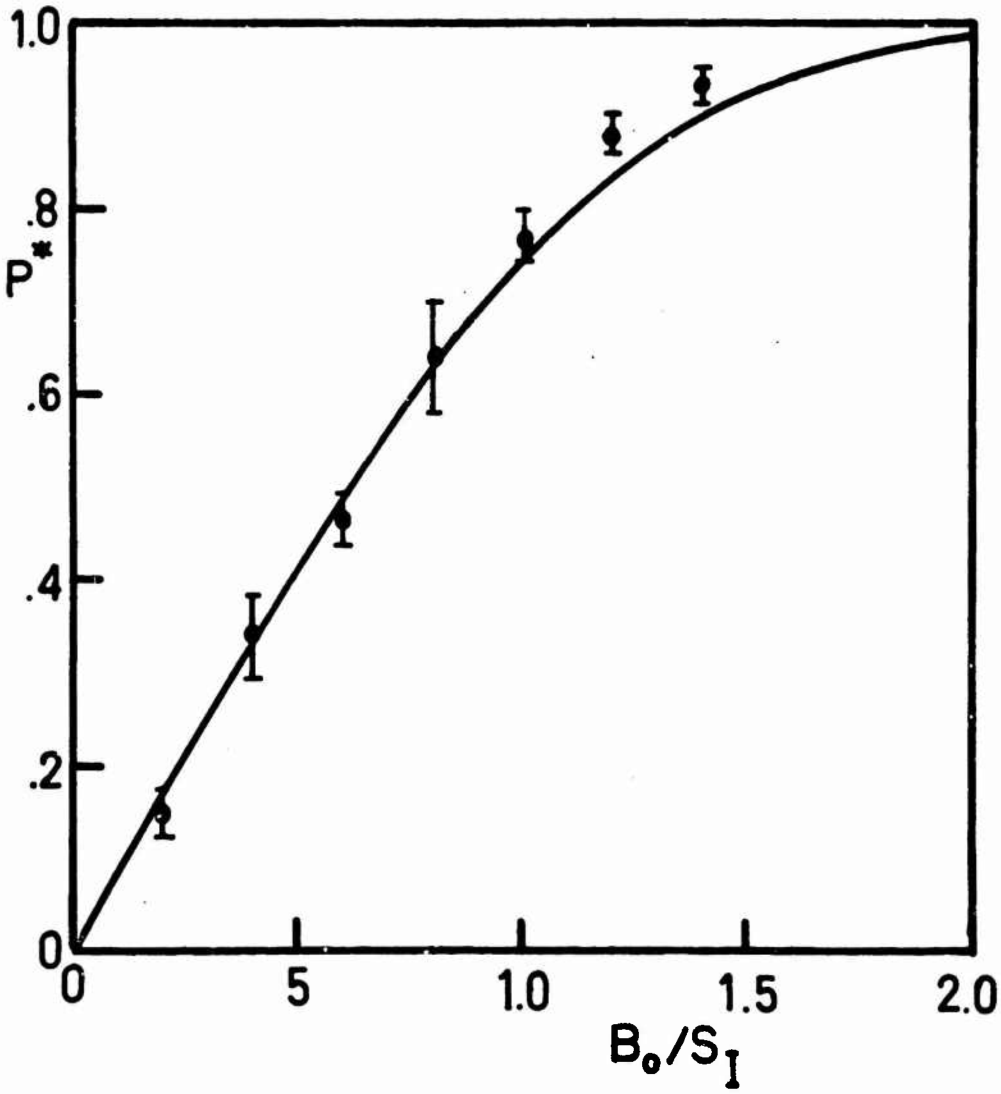

Figure 11: P* (B_o/S_I) in Case XI. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

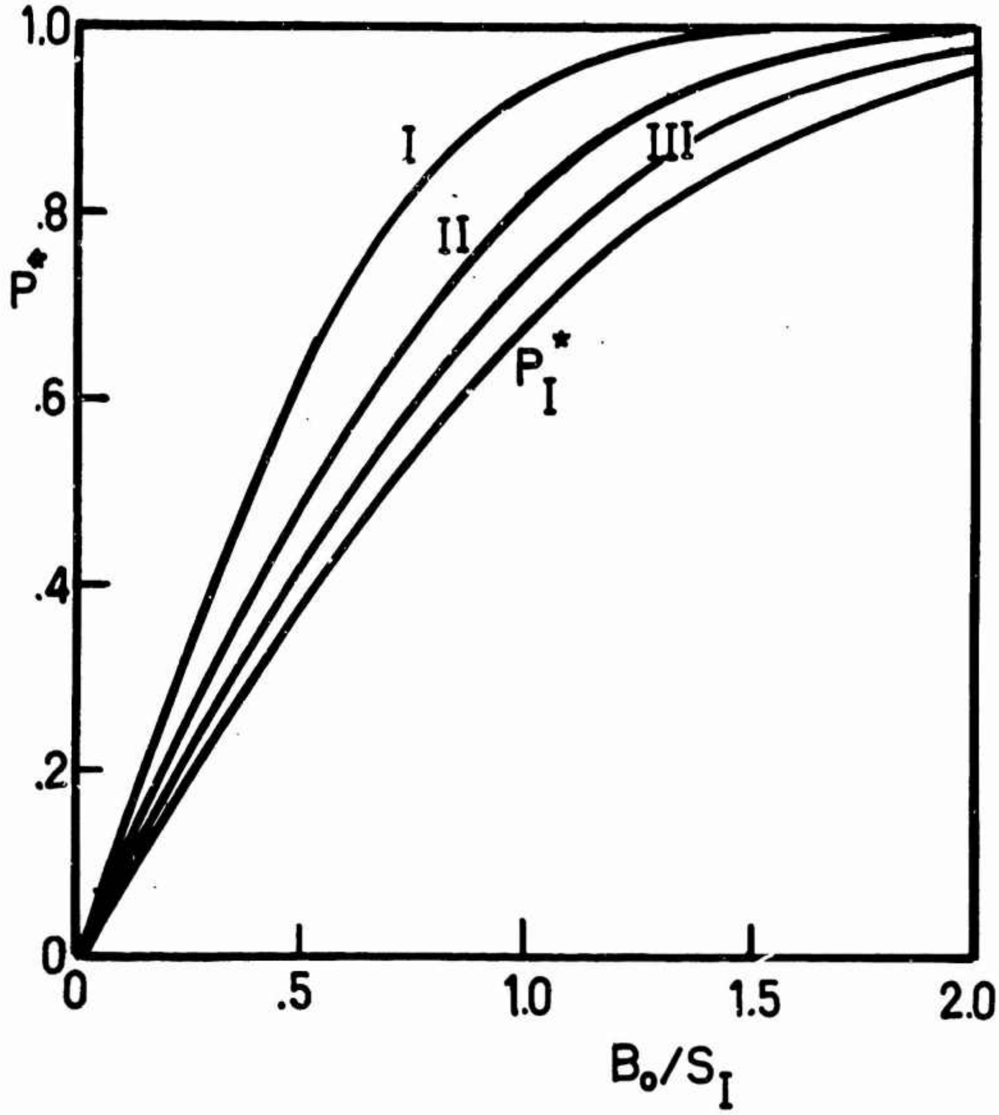

Figure 12: Comparison of P* (B_O/S_I) for Cases (I), (II), (III), using moment prediction, and P^*_I.

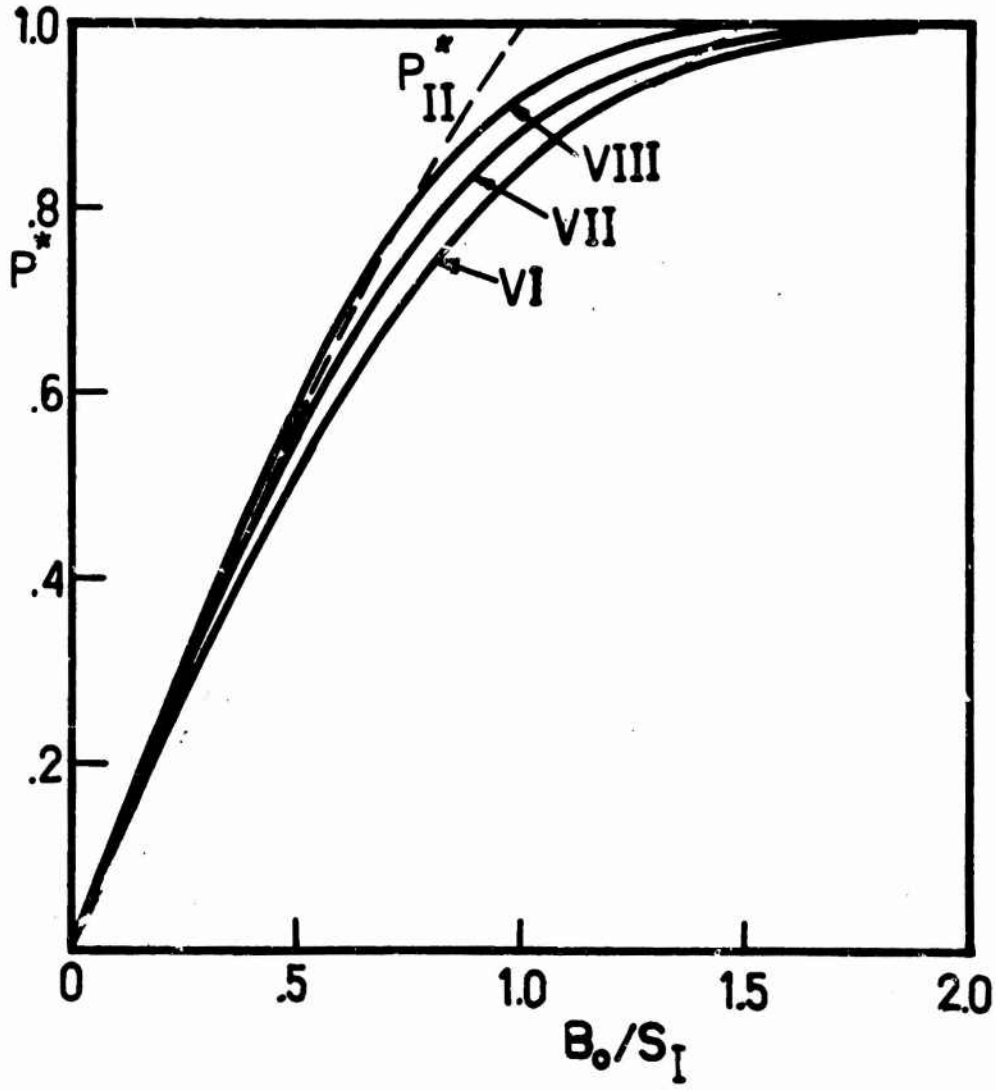

Figure 13: Comparison of P* (B_o/S_I) for Cases (VI), (VII), (VIII), using moment prediction, and P^*_{II}.

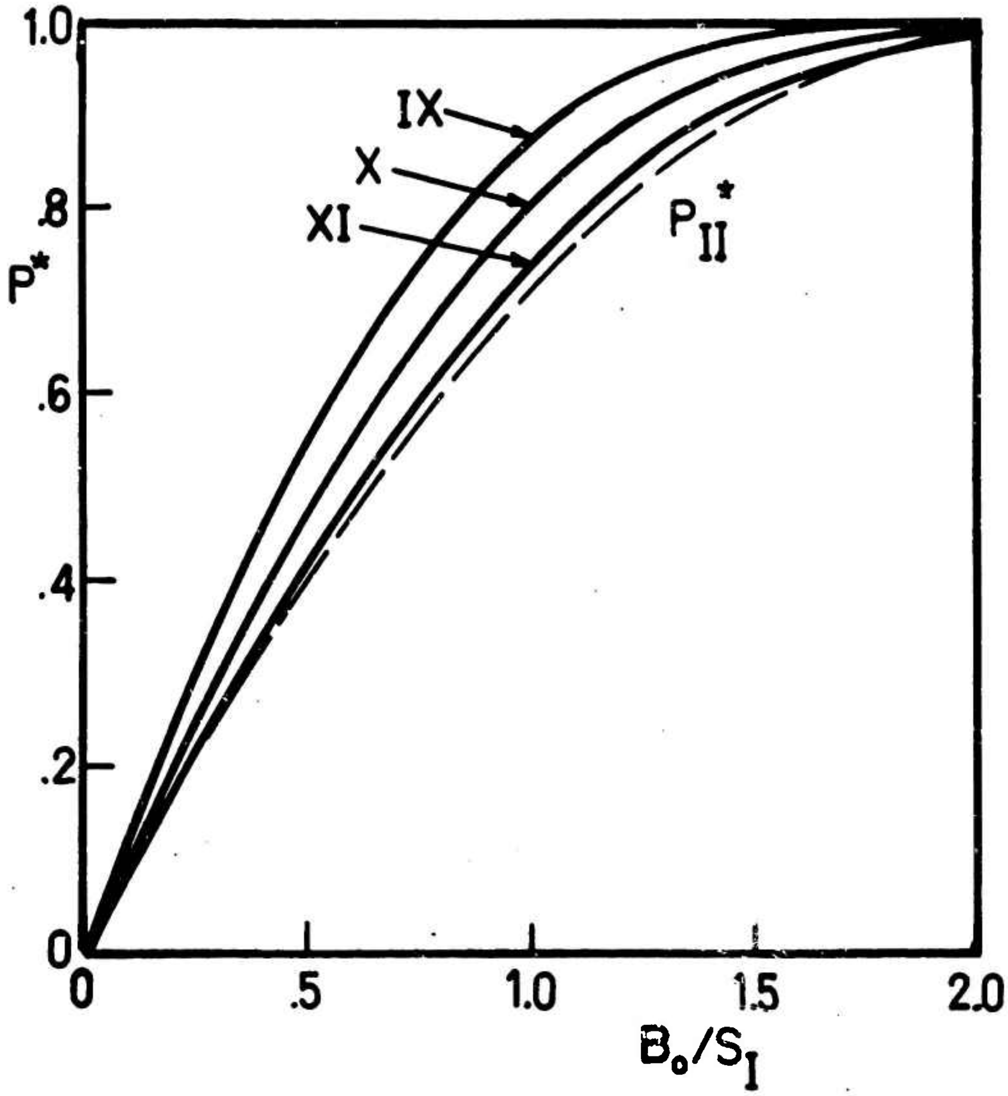

Figure 14: Comparison of P* (B_O/S_I) for Cases (IX), (X), (XI), using moment prediction, and P^*_{II}.

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13. ABSTRACT

This report describes an investigation of how errors in components of an assembly can affect its performance. In particular the report deals with the situation, uncommon in engineering practice, where the output tolerance of the assembly may be violated even though the tolerances on the components are all met. This situation is analyzed to estimate the probability that the output tolerance will be satisfied given that the component tolerances are met. Three methods are described for estimating this probability, their results are compared in a number of cases, and a best method is chosen. Several simple rules, suitable for preliminary estimates, are also given. An example is worked out showing a simple application of the method.

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Tolerances (mechanics)	8		6,7			
Design	9					
Tests	9					
Equipment	9					
Components			6			
Output			7			
Performance			7			
Probability			8,9			
Estimating			8			
Methods			8			